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Variational iteration method for twelfth-order boundary-value problems

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ABSTRACT

In this paper, He's variational iteration method is applied to solve twelfth-order boundary-value problems. The numerical results obtained with minimum amount of computation are compared with the exact solutions to show the efficiency of the method. The results show that the variational iteration method is of high accuracy, more convenient and efficient for solving high-order boundary-value problems.

1. Introduction

We consider the twelfth-order boundary-value problem of the type:

$$y^{xii}(x) + f(x)y = g(x) \quad a < x < b \tag{1}$$

$$y^{2n}(0) = \alpha_{2n}, \quad y^{2n}(1) = \beta_{2n}, n = 0, 1, \dots, 5 \tag{2}$$

where α_{2n} and β_{2n} $(n = 0, 1, \dots, 5)$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$. This type of boundary-value problem arises in an infinite horizontal layer of fluid is heated from below, with the assumption that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modeled by tenth-order boundary-value problems, when instability sets in as over stability, it is modeled by twelfth-order boundary-value problems (see Chandrasekhar [1]). Agarwal's book [2] contains theorems which detail the conditions for existence and uniqueness of solutions of the twelfth-order boundary-value problems. Different numerical methods have been proposed by various authors for twelfth-order boundary-value problems. To mention a few, finite difference method [3,4], modified decomposition method [5], twelfth degree splines [6] and thirteen degree splines [7].

In this paper, we aim to apply the variational iteration method proposed by He [8–14] to twelfth-order boundary-value problems. This method has been employed to solve a large variety of linear and nonlinear problems such as the KdV, the K(2, 2), the Burgers and the cubic Boussinesq equations [23], Korteweg–de vries (KdV) and the modified Korteweg–de vries equations [16], nonlinear differential equations of fractional order [17], KdV Burgers equations and Lax's seventh-order KdV equations [18], generalized Burgers–Fisher and Burgers equations [19], Helmholtz equation [20], semi-linear inverse parabolic equation [15], linear fractional differential equation [21], nonlinear partial differential equations [22] etc.

2. Variational iteration method

According to the variational iteration method [8–14], we can construct a correction functional for twelfth-order boundary-value problem (1)–(2) in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda[y_n^{xii}(s) + f(s)y_n(s) - g(s)]\mathrm{d}s \tag{3}$$

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where λ is a general Lagrange multiplier which can be identify via variational theory [8–10]. Making $y_{n+1}(x)$ stationary with respect to $y_n(x)$, we can identify the Lagrange multiplier, which reads

$$\lambda = \frac{(s-x)^{11}}{11!}. \tag{4}$$

Example 1. First, we consider the following BVP [7]

$$y^{xii}(x) + xy(x) + (120 + 23x + x^3)e^x = 0 \quad 0 < x < 1 \tag{5}$$

with boundary conditions

$$\left.\begin{array}{l} y(0) = 0 = y(1), y'(0) = 1, y'(1) = -e, y''(0) = 0, \\ y''(1) = -4e, y'''(0) = -3, y'''(1) = -9e, y^{iv}(0) = -8, \\ y^{iv}(1) = -16e, y^{v}(0) = -15, y^{v}(1) = -25e \end{array}\right\}. \tag{6}$$

According to (3), we have the following iteration formulation

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^{11}}{11!}[y_n^{xii}(s) + sy_n(s) + (120 + 23s + s^3)e^s]\mathrm{d}s. \tag{7}$$

Now, we assume that an initial approximation has the form

$$y_0(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 + jx^9 + kx^{10} + lx^{11} \tag{8}$$

where $a, b, c, d, e, f, g, h, i, j, k$ and l are unknown constants to be further determined. By iteration formula (7), we have the following first-order approximation

$$\begin{aligned} y_1(x) &= y_0(x) + \int_0^x \frac{(s-x)^{11}}{11!}[y_0^{xii}(s) + sy_0(s) + (120 + 23s + s^3)e^s]\mathrm{d}s \\ &= -\frac{l}{1295295050649600}x^{24} - \frac{k}{647647525324800}x^{23} - \frac{j}{309744468633600}x^{22} \\ &\quad - \frac{i}{140792940288000}x^{21} - \frac{h}{60339831552000}x^{20} - \frac{g}{24135932620800}x^{19} \\ &\quad - \frac{f}{8892185702400}x^{18} - \frac{e}{2964061900800}x^{17} - \frac{d}{871782912000}x^{16} - \frac{c}{217945728000}x^{15} \\ &\quad - \frac{b}{43589145600}x^{14} - \frac{a}{6227020800}x^{13} + \left(l + \frac{13}{5702400}\right)x^{11} + \left(k + \frac{1}{72576}\right)x^{10} \\ &\quad + \left(j - \frac{1}{40320}\right)x^9 + \left(i - \frac{23}{10080}\right)x^8 + \left(h - \frac{41}{1008}\right)x^7 + \left(g - \frac{59}{120}\right)x^6 + \left(f - \frac{109}{24}\right)x^5 \\ &\quad + \left(e - \frac{98}{3}\right)x^4 + \left(d - \frac{359}{2}\right)x^3 + (c - 715)x^2 + (b - 1849)x + (a - 2340) \\ &\quad + (2340 - 491x + 36x^2 - x^3)e^x. \end{aligned}$$

Incorporating the boundary conditions, Eq. (6), into $y_1(x)$, and solving the system simultaneously, we obtain

$$\begin{aligned} &a = 0, \quad b = 1, \quad c = 0, \quad d = -0.5, \quad e = -1/3, \quad f = -1/8, \quad g = -0.03333333362911, \\ &h = -0.00694444319430, \quad i = -0.00119047835708, \quad j = -0.00017360919905, \\ &k = -0.00002204671105, \quad l = -0.00000248000384. \end{aligned}$$

We, therefore obtain the following first-order approximate solution

$$\begin{aligned} y_1(x) = &\frac{0.0000024899384}{1295295050649600}x^{24} + \frac{0.00002204671105}{647647525324800}x^{23} \\ &+ \frac{0.00017360919905}{309744468633600}x^{22} + \frac{0.001190478357105}{140792940288000}x^{21} + \frac{0.00694444319430}{60339831552000}x^{20} \\ &+ \frac{0.03333333362911}{24135932620800}x^{19} + \frac{1/8}{8892185702400}x^{18} + \frac{1/3}{2964061900800}x^{17} \\ &+ \frac{1/2}{87178291200}x^{16} - \frac{1}{43589145600}x^{14} - 2.00261976924803\text{e}{-}007x^{11} \end{aligned}$$

Table 1
Comparison of the first-order approximate solution, Eq. (9), with the exact solution.

x	VIM solution $y_1(x)$	Exact solution	Absolute error
0.0	0	0	0
0.1	0.09946538262776	0.09946538262681	9.524186994624984e−013
0.2	0.19542444130575	0.19542444130563	1.245392677873269e−013
0.3	0.28347034959130	0.28347034959096	3.351208199831035e−013
0.4	0.35803792743337	0.35803792743390	5.379585665821196e−013
0.5	0.41218031767423	0.41218031767503	8.044676036433884e−013
0.6	0.43730851209258	0.43730851209372	1.140698646651117e−012
0.7	0.42288806856919	0.42288806856880	3.933520176246930e−013
0.8	0.35608654855867	0.35608654855879	1.230682222796986e−013
0.9	0.22136428000330	0.22136428000413	8.254508188088039e−013
1.0	3.268502189e−013	0	3.268502189413308e−013

$$
\begin{aligned}
&-8.268051438007055e-006x^{10}-1.984107863515873\mathrm{e}{-}004x^{9}\\
&-0.00347222438883x^{8}-0.04761904636890x^{7}-0.52500000029578x^{6}\\
&-4.66666666666667x^{5}-33x^{4}-180x^{3}-715x^{2}-1848x-2340\\
&+(2340-491x+36x^{2}-x^{3})e^{x}.
\end{aligned}
\tag{9}
$$

The exact solution of the above differential equations is

$$y(x)=x(1-x)e^{x}.$$

Comparison of the first-order approximate solution with the exact solution is tabulated in Table 1 showing a remarkable agreement.

Example 2. Next, we consider the following BVP [7]

$$y^{xii}(x)-y(x)+12(2x\cos x+11\sin x)=0\quad -1<x<1 \tag{10}$$

$$
\begin{aligned}
&y(-1)=0=y(1),\qquad y'(-1)=y'(1)=2\sin(1),\\
&y''(-1)=-y''(1)=-4\cos(1)-2\sin(1),\\
&y^{(iii)}(-1)=y^{(iii)}(1)=6\cos(1)-6\sin(1),\\
&y^{(vi)}(-1)=-y^{(iv)}(1)=8\cos(1)+12\sin(1),\\
&y^{(v)}(-1)=y^{(v)}(1)=-20\cos(1)+10\sin(1).
\end{aligned}
\tag{11}
$$

According to (3), we have the following iteration formulation

$$y_{n+1}(x)=y_n(x)+\int_0^x\frac{(s-x)^{11}}{11!}[y_n^{xii}(s)-y_n(s)+12(2s\cos s+11\sin s)]\mathrm{d}s. \tag{12}$$

Now, we assume that an initial approximation has the form

$$y_0(x)=a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+ix^8+jx^9+kx^{10}+lx^{11} \tag{13}$$

where $a, b, c, d, e, f, g, h, i, j, k$ and l are unknown constants to be further determined. By iteration formula (12), we have the following first-order approximation

$$
\begin{aligned}
y_1(x)=\;&\frac{l}{647647525324800}x^{23}+\frac{k}{309744468633600}x^{22}+\frac{j}{140792940288000}x^{21}\\
&+\frac{i}{60339831552000}x^{20}+\frac{h}{24135932620800}x^{19}+\frac{g}{8892185702400}x^{18}\\
&+\frac{f}{2964061900800}x^{17}+\frac{e}{871782912000}x^{16}+\frac{d}{217945728000}x^{15}\\
&+\frac{c}{43589145600}x^{14}+\frac{b}{6227020800}x^{13}+\frac{a}{479001600}x^{12}\\
&+\left(l-\frac{1}{369600}\right)x^{11}+\left(j+\frac{1}{6048}\right)x^{9}+\left(h-\frac{1}{420}\right)x^{7}+\left(f-\frac{3}{10}\right)x^{5}+a\\
&+(b-132)x+cx^2+(d+14)x^3+ex^4+gx^6+ix^8+kx^{10}+156\sin x-24x\cos x.
\end{aligned}
$$

Incorporating the boundary conditions, Eq. (11), into $y_1(x)$, and solving the system simultaneously, we obtain

$$a=-0.277555755576842759\mathrm{e}{-}16,\qquad b=-0.999999999999996891,$$

Table 2a
Comparison of the first-order approximate solution, Eq. (14), with the exact solution.

x	Exact solution	VIM solution $y_1(x)$	Absolute error
−1.0	0	−0.932587340685e−14	0.93258734068513e−14
−0.9	0.1488321128292218	0.1488321128292236	0.1776356839400e−14
−0.8	0.2582481927238280	0.2582481927238152	0.1282307593442e−13
−0.7	0.3285510204912224	0.3285510204912206	0.1831867990631e−14
−0.6	0.3613711829728227	0.3613711829728373	0.1465494392505e−13
−0.5	0.3595691539531523	0.3595691539531660	0.1376676550535e−13
−0.4	0.3271114075392664	0.3271114075392686	0.1332267629550e−14
−0.3	0.2689233880618189	0.2689233880618338	0.1482147737874e−13
−0.2	0.1907225575632587	0.1907225575632596	0.9436895709313e−15
−0.1	0.0988350824803598	0.0988350824803585	0.5551115123125e−16
0	0	−.277555755576e−16	0.27755575557684e−16

Table 2b
Comparison of the first-order approximate solution, Eq. (14), with the exact solution.

x	Exact solution	VIM solution $y_1(x)$	Absolute error
0.0	0	−2.77555755576e−017	2.7755575557684e−017
0.1	−0.0988350824803598	−0.0988350824803584	1.3877787807814e−015
0.2	−0.1907225575632587	−0.1907225575632595	8.3266726846886e−016
0.3	−0.2689233880618190	−0.2689233880618338	1.6542323066914e−014
0.4	−0.3271114075392664	−0.3271114075392686	2.1649348980190e−015
0.5	−0.3595691539531523	−0.3595691539531660	1.3766765505351e−014
0.6	−0.3613711829728227	−0.3613711829728373	2.2204460492503e−015
0.7	−0.3285510204912224	−0.3285510204912206	1.6098233857064e−014
0.8	−0.2582481927238281	−0.2582481927238155	1.2601031329495e−014
0.9	−0.1488321128292218	−0.1488321128292238	1.9984014443252e−015
1.0	0	0.9325873406851e−14	9.3258734068513e−015

$$c = -0.971445158559381916\text{e}{-16}, \qquad d = 1.166666666666666667,$$
$$e = 0.100845239619727242\text{e}{-15}, \qquad f = -0.175000000000001599,$$
$$g = -0.345504587738423279\text{e}{-16}, \qquad h = 0.853174603174756963\text{e}{-2},$$
$$i = 0.955915575530589730\text{e}{-17}, \qquad j = -0.201168430335752597\text{e}{-3},$$
$$k = -0.132768981572782405\text{e}{-17}, \qquad l = 0.278078403089638659\text{e}{-5}.$$

We, therefore obtain the following first-order approximate solution

$$\begin{aligned}
y_1(x) = {} & \frac{0.278078403089638659e-5}{647647525324800}x^{23} - \frac{0.132768981572782405e-17}{309744468633600}x^{22} \\
& - \frac{0.201168430335752597\text{e}{-3}}{140792940288000}x^{21} + \frac{0.955915575530589730\text{e}{-17}}{60339831552000}x^{20} \\
& + \frac{0.853174603174756963\text{e}{-2}}{24135932620800}x^{19} - \frac{0.345504587738423279\text{e}{-16}}{8892185702400}x^{18} \\
& - \frac{0.175000000000001599}{2964061900800}x^{17} + \frac{0.100845239619727242\text{e}{-15}}{871782912000}x^{16} \\
& + \frac{1.166666666666666667}{217945728000}x^{15} - \frac{0.971445158559381916\text{e}{-16}}{43589145600}x^{14} \\
& - \frac{0.999999999999996891}{6227020800}x^{13} - \frac{0.277555755576842759\text{e}{-16}}{479001600}x^{12} \\
& + 0.751563252686809709\text{e}{-7}x^{11} - 0.451134989999223829e - 4x^{9} \\
& + 0.615079365079518846\text{e}{-2}x^{7} - 0.475000000000001587x^{5} \\
& - 0.277555755576842759\text{e}{-16} - 133x - 0.971445158559381916\text{e}{-16}x^{2} \\
& + 15.166666666666666667x^{3} + 0.100845239619727242\text{e}{-15}x^{4} \\
& - 0.3455045877384279\text{e}{-16}x^{6} + 0.955915575530589730\text{e}{-17}x^{8} \\
& - 0.132768981572782405\text{e}{-17}x^{10} + 156\sin x - 24x\cos x.
\end{aligned} \tag{14}$$

The exact solution of the above differential equations is $y(x) = (x^2 - 1)\sin x$. Comparison of the first-order approximate solution with the exact solution is tabulated in Tables 2a and 2b showing a remarkable agreement.

3. Conclusion

Variational iteration method is remarkably effective for solving twelfth-order boundary-value problems. One iteration is enough to obtain very high accurate solution. This method is very powerful, efficient and promoting, which will be certainly found wide applications.

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